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# STABILITY OF COMPLICATED ROTOR-BEARING SYSTEM BY OVERLAPPING DECOMPOSITION–AGGREGATION METHOD

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The whole and the part are general attributes of nature. By means of an overlapping decomposition–aggregation method, this paper proposed a way and procedure to show how a system is decomposed and then aggregated into a new expanded system. In this expanded system, the original one is viewed and treated as two levels: the whole system and its subsystems. We hence find a methodology that can provide us with the information about a system and its specified subsystems, and the relationship between the whole and the parts. This paper shows an application of the standpoint and the method to stability problems of a complicated rotor-bearing system. A numerical example is also given to show the uniqueness and superiority of the methodology. It is believed that the proposed method provides us with a powerful and flexible tool to analyze the dynamic behavior for the complicated rotor-bearing system. It can also be used in other similar compound dynamic systems.



## 1. INTRODUCTION

The complicated rotor-bearing system here denotes a rotor dynamic system with at least three features: (1) it is supported by hydrodynamic lubricated bearings, (2) the dimension of the system can be very large, and (3) it runs above at least its first critical speed. The role of the hydrodynamic lubricated bearing is emphasized because it is not only the source of damping for the system but also the cause of instability and its dynamic behavior is in essence non-linear. Therefore, the contradiction between the stability and instability emerges from time to time. To some extent, the stability of this kind of system should first be emphasized and solved.

However, in Rotodynamics the role of the hydrodynamic lubricated bearing is often simplified as a non-linear spring and damping element, while in Tribology, the bearing and a simplified rotor are thought to constitute a complete dynamic system. Since the problems to be solved for this kind of system are traditionally focused on searching the behavior of the overall system, the analysis methods aiming at the entire system such as the mode superposition method or the eigenvalue analysis method are widely used. The amount of literature on this topic is abundant. However, along with the progress in the understanding of the dynamic characteristics of hydrodynamic lubricated bearings and the requirement for gaining an insight into the stability of the system, the following questions are posed:

(1) How does the dynamic behavior of the bearing subsystem affect that of the whole system?

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(2) What are the interconnections between the system and its bearing subsystems, and which one most influences the behavior of the entire system?

Obviously, according to the problems to be solved, the focus of the studies should be placed not only on the whole system but also its subsystems. However, these tasks are difficult to tackle by the traditional analysis methods mentioned before. In the search for a proper analysis method, the stability theory of large-scale dynamic systems provides us with an alternative way to tackle the problems. By adopting the classical decomposition–aggregation method, Zhang and Zhu [1, 2] decomposed the rotor-bearing system into bearing and rotor subsystems and studied its stability and dynamic performances. However, the analysis results often tend to be conservative. Therefore, in the application of the method onto the rotor-bearing system in which the rotor and the bearings are in fact strongly coupled to each other, there exists the difficulty to handle the interconnection between the decomposed subsystems. The superiority of the method does not fully show up.

On the other hand, the gist of the classical decomposition–aggregation method is that the information on the system is obtained by means of its subsystems. However, in many practical engineering problems like the rotor-bearing system, the relationship of the system with its subsystems is to be investigated. For instance, we want to know the relationship between the dynamic behavior of the whole rotor-bearing system and that of some bearing subsystems. Therefore, a decomposing–aggregation of new significance is needed which is able to retain the advantages of the classical method and overcome its drawbacks when applied to the rotor-bearing system.

By adopting the idea of an overlapping decomposition–aggregation approach suggested by Ikeda and Siljak [3–5], the author developed the method and analyzed with it the dynamic relationship between the system and its subsystems. According to the problems to be investigated, the author views the rotor-bearing system as comprising two levels, namely, the whole system and the subsystems, no matter what kind of couplings the system has between its interconnected subsystems. The relationship of dynamic behavior between the system and its decomposed subsystems is stressed and studied, and the dynamic status of the system and its decomposed subsystems can be obtained simultaneously in this new method. The paper presents in detail how the author's standpoint and goal can be realized. The study shows that the proposed methodology successfully achieves the required goals and results. The approach overcomes the shortcomings of the classical decomposition–aggregation method in the study of the stability of a rotor-bearing system. It is believed that the methodology is also adapted to similar dynamic issues from similar compound dynamic systems.

## 2. STABILITY OF ROTOR-BEARING SYSTEM

In Tribology the simplest mechanical model of a hydrodynamic lubricated bearing can be depicted by a rigid rotor acting on a hydrodynamic force generated within the lubricating oil, as illustrated in Figure 1. In general, the dynamic force generated from the hydrodynamic oil film of bearing is in essence non-linear. The stability analysis for this kind of system is therefore of theoretical and practical significance. On the other hand, the dynamic behavior of the rotor-bearing system strongly depends on the revolution speed of the rotor. For some range of the rotor speeds, Zhang [6, 7] proposed a mathematical model for this kind of fluid dynamic force:

$$\mathbf{f}(\mathbf{X}_b) = \mathbf{A}_b\mathbf{X}_b + \mathbf{r}(\mathbf{X}_b), \tag{1}$$

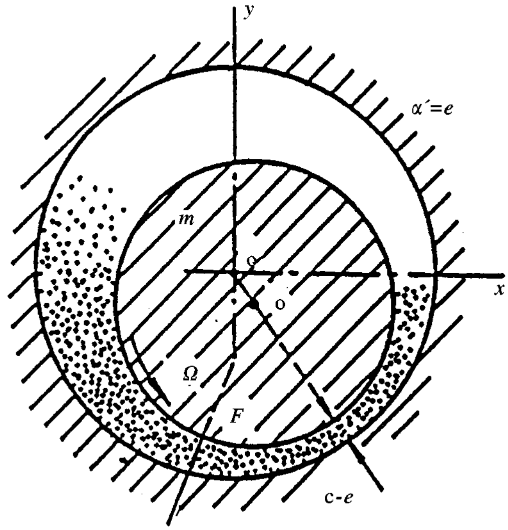


Figure 1. Configuration of hydrodynamic lubricated bearing system.

where $\mathbf{X}_b = (x\ y\ \dot{x}\ \dot{y})^{\mathrm{T}}$. $\mathbf{A}_b \in \mathbf{R}^{4\times 4}$ is a constant matrix, and $\mathbf{r}(\mathbf{X}_b)$ is non-linear function of $\mathbf{X}_b$ with orders equal to or higher than 2.

Therefore, the stability of the above rotor-bearing dynamic system in the vicinity of its equilibrium $\mathbf{X}_b = 0$ can be evaluated by its linearized system according to Liapunov's first approximation theorem [8]

$$\mathbf{M}_b\ddot{\mathbf{x}} + \mathbf{C}_b\dot{\mathbf{x}} + (\mathbf{S}_b + \mathbf{F}_b)\mathbf{x} = 0, \tag{2}$$

where $\mathbf{x} = (x\ y)^{\mathrm{T}}$, $\mathbf{S}_b = \mathbf{S}_b^{\mathrm{T}}$, $\mathbf{F}_b = -\mathbf{F}_b^{\mathrm{T}}$ and $\mathbf{M}_b > 0$.

As we know, because of the existence of the constant matrix $\mathbf{F}_b$, the famous KTC criterion of the stability theory fails to evaluate its stability. Alternatively, the Liapunov second method will be used here to study the stability described by equation (2). And understandably, for some special cases when the non-linear part of equation (1) can be accessed explicitly and the Liapunov function can be constructed, the stability problems of some non-linear systems can also be tackled by the method [9]. But, the construction of the general Liapunov-type function for a general non-linear dynamic system is not usually an easy task.

On the other hand, when a rotor-bearing system operates above its first critical speed, the rotor cannot be treated as a rigid one. By means of the finite element method or the transfer matrix method, etc., modern Rotordynamics can set-up $n$-dimensional equations of motion for a rotor system with even complicated configurations. The first approximation of the system can have the form

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{P}(\mathbf{t}), \quad \mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathbf{R}^{n\times n}, \quad \mathbf{x} \in \mathbf{R}^n \times \mathbf{J}_+, \tag{3}$$

where $\mathbf{K} = \mathbf{S} + \mathbf{F}$ denotes the stiffness matrix of the system, $\mathbf{P}(t)$ the external force vector and $\mathbf{x}(t)$ the displacement vector of the system with $\mathbf{J}_+ \in [0, +\infty)$.

As a Liapunov function for a linearized dynamic system is easily constructed, the $n$-dimensional linearized rotor-bearing system with $s-1$ $(s > 2)$ hydrodynamic lubricated bearings is accordingly taken as an example here to explain the methodology. Obviously, the stability of the system described by equations (3) can be determined by the stability of its equilibrium $\mathbf{x} \equiv 0$, i.e.,

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0, \tag{4}$$

and equation (4) can be transformed and expressed in state space as

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X}, \tag{5}$$

where

$$\mathbf{X} = \begin{Bmatrix} \dot{\mathbf{x}} \\ \mathbf{x} \end{Bmatrix}, \quad \mathbf{X} \in \mathbb{R}^N \times \mathbf{J}_+ . \tag{6}$$

$N = 2n$ and the system matrix $\mathbf{A}$ is only related to $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ of the original system and can be expressed in state space as

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{O} \end{bmatrix}, \quad \mathbf{A} \in \mathbb{R}^{N \times N}, \tag{7}$$

where $\mathbf{I}$ denotes the unit matrix, and $\mathbf{M} > 0$.

According to the Liapunov stability theory [8], the stability of motion for the linear system of equation (5) can be evaluated by the Liapunov second method. This is, if there exists a symmetric and positive matrix $\mathbf{H}$ satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{H} + \mathbf{H}\mathbf{A} = -\mathbf{I} \tag{8}$$

the system is exponentially stable.

For the stable system, the degree of its stability status can be estimated by the stability degree defined by

$$\alpha \leqslant \min\left[ -\frac{\dot{V}(\mathbf{X}, t)}{V(\mathbf{X}, t)} \right], \tag{9}$$

where $V(\mathbf{X}, t)$ denotes the Liapunov function. For the linearized system, the Liapunov function can be chosen as

$$V(\mathbf{X}, t) = \mathbf{X}^{\mathrm{T}}\mathbf{H}\mathbf{X}, \tag{10}$$

and in this case, the stability degree of the system is

$$\alpha = \frac{1}{\lambda_M(\mathbf{H})}, \tag{11}$$

where $\lambda_M(\mathbf{H})$ is the maximum eigenvalue of the symmetric and positive Liapunov matrix $\mathbf{H}$.

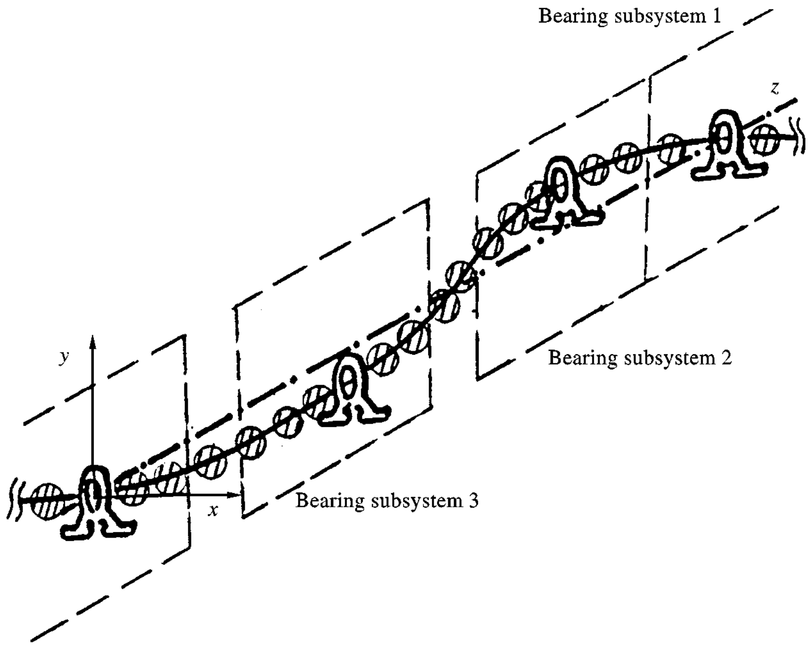


Figure 2. Configuration of complicated rotor-bearing systems with lumped-mass model.

Stability degree is also a measure of the speed of a stable system approaching its equilibrium. It plays a similar role to the damping ratio or decay rate in vibration theory. Therefore, this characteristic index expresses a dynamic performance of the system as well.

## 3. ROTOR-BEARING SYSTEM AND ITS BEARING SUBSYSTEMS

In order to investigate the relationship between a system and its subsystems, the subsystems to be studied should first be specified and abstracted from this system. The author think there exists at least one rule to decompose a subsystem from the system. That is, a subsystem abstracted from a system must constitute a complete dynamic entity.

Figure 2 illustrates intuitively the concept of the whole system and its subsystems. The simple lumped-mass model of the rotor-bearing system is used here, in which the parts of the system included in a dashed-line frame demonstrate the bearing subsystems. The methodology will be systematically explained in several procedures of the following sections.

### 3.1. SUBSYSTEMS BY OVERLAPPING DECOMPOSITION

The subsystems denoted here have a general meaning. They denote not only the subsystems with practical entities from the realistic system, but also those of only mathematical significance corresponding to an expanded system by an overlapping-decomposition procedure.

3.1.1. *Procedure* 1. *Identification of bearing subsystems*

At first, the identification of the bearing subsystems from an original rotor-bearing system should be made. Due to the structure of the rotor-bearing system, as illustrated in Figure 2, the system matrix $\mathbf{A}$ of equation (5) can be rearranged as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 & \mathbf{A}_{1s} \\ 0 & \mathbf{A}_2 & \cdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & \cdots \\ 0 & \cdots & \cdots & \mathbf{A}_{s-1} & \mathbf{A}_{s-1s} \\ \mathbf{A}_{s1} & \cdots & \cdots & \mathbf{A}_{ss-1} & \mathbf{A}_s \end{bmatrix}, \tag{12}$$

where $\mathbf{A}_i$, $i = 1, 2, \ldots, s-1$ represent the $i$th bearing subsystem and

$$\mathbf{A}_i = \begin{bmatrix} -\mathbf{M}_i^{-1}\mathbf{C}_i & -\mathbf{M}_i^{-1}\mathbf{K}_i \\ \mathbf{I}_i & 0 \end{bmatrix}, \quad i = 1, 2, \ldots, s-1. \tag{13}$$

It can be seen that subsystem matrices $\mathbf{A}_i$ satisfy the rule described before. Therefore, the isolated bearing subsystem can be expressed by

$$\dot{\mathbf{x}}_i = \mathbf{A}_i \mathbf{x}_i, \quad i = 1, 2, \ldots, s-1. \tag{14}$$

The system matrix $\mathbf{A}$ can then be expressed as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \tag{15}$$

where

$$\mathbf{A}_{11} = \begin{bmatrix} \mathbf{A}_1 & & & 0 \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ 0 & & & \mathbf{A}_{s-1} \end{bmatrix}, \tag{16}$$

$$\mathbf{A}_{12} = \begin{bmatrix} \mathbf{A}_{1s} \\ \vdots \\ \mathbf{A}_{s-1s} \end{bmatrix}, \quad \mathbf{A}_{21} = [\mathbf{A}_{s1}\, \mathbf{A}_{s2} \cdots \mathbf{A}_{ss-1}], \tag{17}$$

$$\mathbf{A}_{22} = \mathbf{A}_s. \tag{18}$$

Therefore, the equation of motion of equation (5) can be expressed in state space as

$$\begin{bmatrix} \dot{\mathbf{X}}_1 \\ \dot{\mathbf{X}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}. \tag{19}$$

The identification of the bearing subsystems obviously depends on the researcher's experience although the purely mathematical decomposition is theoretically available. The author suggests a principle for the identification or decomposition of a bearing subsystem from the rotor-bearing system: the bearing subsystem should be identified by its loading distribution.

### 3.1.2. *Procedure* 2. *Overlapping decomposition*

According to the methodology of overlapping decomposition–aggregation [3], the original system should be expanded into a new system in line with the subsystems to be studied. As we treat the entire system as one of two levels, the original rotor-bearing system is required to be included in the expanded system. To this end, suppose the new state variable $\mathbf{Z}$:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}, \tag{20}$$

such that

$$\mathbf{Z} = \mathbf{TX}, \tag{21}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}, \tag{22}$$

where

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_1 & 0 \\ \mathbf{I}_1 & 0 \\ 0 & \mathbf{I}_2 \end{bmatrix}. \tag{23}$$

Here, $\mathbf{I}_1$ and $\mathbf{I}_2$ denote the unit matrix with the dimension corresponding to the related subspaces respectively.

Thus, the new system is expanded from the original system as

$$\dot{\mathbf{Z}} = \bar{\mathbf{A}}\mathbf{Z} \qquad \bar{\mathbf{A}} \in \mathbf{R}^{N \times N}, \quad \mathbf{Z} \in \mathbb{R}^{\bar{N}} \times \mathbf{J}_+, \tag{24}$$

where $\bar{N} = 2n_1 + n_2$, $n_1$ and $n_2$ denote the dimensions of the subspace $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively, and the new system matrix $\bar{\mathbf{A}}$:

$$\bar{\mathbf{A}} = \mathbf{TAT}^{-1} + \mathbf{B}, \tag{25}$$

where $\mathbf{T}^{-1}$ denotes the generalized inverse matrix of $\mathbf{T}$ and matrix $\mathbf{B}$ is the complemented matrix related with system matrix $\bar{\mathbf{A}}$.

If we take the generalized inverse matrix $\mathbf{T}^{-1}$ in equation (25) as the pseudo-matrix, i.e.,

$$\mathbf{T}^{-1} = \mathbf{T}^{I} = (\mathbf{T}^{\mathrm{T}}\mathbf{T})^{-1}\mathbf{T}^{\mathrm{T}},$$
$$= \begin{bmatrix} \frac{1}{2}\mathbf{I}_1 & \frac{1}{2}\mathbf{I}_2 & 0 \\ 0 & 0 & \mathbf{I}_2 \end{bmatrix}, \tag{26}$$

we have

$$\mathbf{TAT}^{1} = \begin{bmatrix} \frac{1}{2}\mathbf{A}_{11} & \frac{1}{2}\mathbf{A}_{11} & \mathbf{A}_{12} \\ \frac{1}{2}\mathbf{A}_{11} & \frac{1}{2}\mathbf{A}_{11} & \mathbf{A}_{12} \\ \frac{1}{2}\mathbf{A}_{21} & \frac{1}{2}\mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}. \tag{27}$$

3.1.3 *Procedure* 3. *Selection of complementary matrix*

According to the requirements, the complementary matrix **B** is chosen so that the subsystems and original overall system in the expanded space can best approach its entity, and the new expansion system is made the most "simplified". We select:

$$\mathbf{B} = \begin{bmatrix} \frac{1}{2}\mathbf{A}_{11} & -\frac{1}{2}\mathbf{A}_{11} & 0 \\ -\frac{1}{2}\mathbf{A}_{11} & \frac{1}{2}\mathbf{A}_{11} & 0 \\ -\frac{1}{2}\mathbf{A}_{21} & \frac{1}{2}\mathbf{A}_{21} & 0 \end{bmatrix}. \tag{28}$$

It can be easily proved that such a selected complementary matrix **B** satisfies the selection criterion of the overlapping decomposition–aggregation method [3, 4], i.e.,

$$\mathbf{T}^{I}\mathbf{B}^{k}\mathbf{T} = 0, \quad k = 1, 2, \ldots, \bar{N}. \tag{29}$$

Obviously, the selection of this complementary matrix also depends on the researcher's experience. However, the advantage of this method is as follows. Substituting equations (27) and (28) into equation (25), the system matrix $\bar{\mathbf{A}}$ of the expansion system becomes

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & 0 & \mathbf{A}_{12} \\ 0 & \mathbf{A}_{11} & \mathbf{A}_{12} \\ 0 & \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}. \tag{30}$$

Referring equations (16) and (17), if we denote

$$\bar{\mathbf{A}}_i = \mathbf{A}_i,$$
$$\bar{\mathbf{A}}_s = \mathbf{A}, \qquad i = 1, 2, \ldots, s-1,$$
$$\bar{\mathbf{A}}_{is} = [0, \mathbf{A}_{is}],$$

the expanded system matrix $\bar{\mathbf{A}}$ can be rewritten as

$$\mathbf{A} = \begin{bmatrix} \bar{\mathbf{A}}_1 & 0 & \cdots & 0 & \bar{\mathbf{A}}_{1s} \\ 0 & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \bar{\mathbf{A}}_{s-1} & \bar{\mathbf{A}}_{s-1s} \\ 0 & \cdots & \cdots & 0 & \bar{\mathbf{A}}_s \end{bmatrix}. \tag{31}$$

It can be recognized now that the original rotor-bearing system $\mathbf{A}$ has been expanded into system $\bar{\mathbf{A}}$. The submatrix $\bar{\mathbf{A}}_s$ of the expanded system is in fact the original overall system.

It should have been recognized that the structure of the expanded system matrix $\bar{\mathbf{A}}$ depends on the selections of the complementary matrix $\mathbf{B}$ and generalized inverse matrix $\mathbf{T}^{-1}$.

### 3.1.4. *Procedure* 4. *Isolated subsystems*

Following the classical decomposition procedure, the expanded dynamic system of equation (24) can be decomposed into

$$\dot{\mathbf{Z}}_i = \bar{\mathbf{A}}_i \mathbf{Z}_i + \sum_{\substack{j=1 \\ i \neq j}}^{s} \bar{\mathbf{A}}_{ij} \mathbf{Z}_j, \quad i = 1, 2, \ldots, s. \tag{32}$$

The isolated subsystems of the expansion system can be obtained [10]:

$$\dot{\mathbf{Z}}_i = \bar{\mathbf{A}}_i \mathbf{Z}_i, \quad i = 1, 2, \ldots, s, \tag{33}$$

and the stability of these isolated subsystems can be evaluated by the criterion and procedure described in section 2 of this paper.

The following explains the meaning of the above procedures.

When Ikeda and Siljak developed the overlapping decomposition to study their own problems, their methodology adopted is the same as that of the classical decomposition–aggregation method, that is, the information of the whole system is obtained by obtaining that of its subsystems. The point of consideration is still placed on the overlapped subsystems.

This study adopts the method of Ikeda and Siljak's overlapping decomposition, but the gist is definitely different. Here, the system is viewed as comprising two levels: the highest level is the original entire system itself and the second level is its subsystems. That is, in line with the problem to be solved, the system is considered to comprise two levels at the same time. But, the result illustrated in equation (30) for the rotor-bearing system by overlapping decomposition–aggregation appears a little unexpected in that it provides us with additional information: not only the information of the different levels of this dynamic system, but also the information of interconnection between them. The detailed study of this interconnection can be referred to in other studies of the author [6].

### 3.2. AGGREGATION OF THE SYSTEM

#### 3.2.1. *Procedure* 5. *Stability of isolated subsystems*

Due to the requirement in engineering applications, it is required that each isolated subsystem presented by equation (33) satisfies the stability requirement (otherwise, re-adjust the subsystem to achieve the required stability). Then, according to the Liapunov stability theory, there exists the Liapunov function for each of these subsystems. Particularly, for the linearized system of our example, we can choose the Liapunov functions for each subsystem of equation (33) as

$$\mathbf{V}_i(\mathbf{Z}_i, t) = (\mathbf{Z}_i^{\mathrm{T}} \mathbf{H}_i \mathbf{Z}_i)^{1/2}, \quad i = 1, 2, \dots, s, \tag{34}$$

where $\mathbf{H}_i$ is the Liapunov matrix.

And it can also be proved that the following inequalities exist [8]:

$$\begin{aligned} &\lambda_m^{1/2}(\mathbf{H}_i)\cdot\|\mathbf{Z}_i\| \leqslant \mathbf{V}_i(\mathbf{Z}_i, t) \leqslant \lambda_M^{1/2}(\mathbf{H}_i)\cdot\|\mathbf{Z}_i\|, \\ &\dot{\mathbf{V}}(\mathbf{Z}_i, t) \leqslant -0{\cdot}5\lambda_M^{1/2}(\mathbf{H}_i)\cdot\|\mathbf{Z}_i\|, \\ &\|[grad\, \mathbf{V}_i(\mathbf{Z}_i, \mathbf{t})]^{\mathrm{T}}\| \leqslant \lambda_M(\mathbf{H}_i)\cdot\lambda_m^{1/2}(\mathbf{H}_i), \\ &\|\mathbf{A}_{ij}\mathbf{Z}_j\| \leqslant \lambda_M^{1/2}(\mathbf{A}_{ij}^{\mathrm{T}}\mathbf{A}_{ij})\cdot\|\mathbf{Z}_j\|, \end{aligned} \qquad i = 1, 2, \dots, s, \tag{35}$$

where $\lambda_m$, $\lambda_M$ denote the minimum and maximum eigenvalues of the matrix.

Now, it can be recognized that the Liapunov direct method itself is in nature an aggregation process as the stability of each subsystem can be represented by a single Liapunov function. Furthermore, the aggregated system can also be evaluated by the Liapunov function. In this study, the vector Liapunov function will be used, and then, based on the comparison principle, the stability of the expanded system can be estimated.

#### 3.2.2. *Procedure* 6. *Aggregation*

To conduct the aggregation of the system, assume that the test function for the aggregation system is

$$V(\mathbf{Z}, t) = \sum_{i=1}^{s} d_i V_i(\mathbf{Z}_i, t), \quad d_i > 0. \tag{36}$$

Taking the time derivative of $V(\mathbf{Z}, t)$ along the solutions of interconnected subsystems equation (32) and substituting equations (34) and (35) into the correspondent variables of $\dot{V}(\mathbf{Z}, t)$, we have

$$\begin{aligned} \dot{\mathbf{V}}(\mathbf{Z}, \mathbf{t}) &= \sum_{i=1}^{s-1} \{d_i \dot{\mathbf{V}}_1(\mathbf{Z}_i, \mathbf{t}) + d_s [\mathbf{grad}\, \mathbf{V}_i(\mathbf{Z}_i, \mathbf{t})]^{\mathrm{T}} \mathbf{A}_{is}\} + d_s \dot{\mathbf{V}}_s(\mathbf{Z}_s, t), \\ &\leqslant \sum_{i=1}^{s-1} \{-0{\cdot}5\, d_i \lambda_M^{-1/2}(\mathbf{H}_i)\cdot\|\mathbf{Z}_1\| + d_s \lambda_M^{1/2}(\mathbf{H}_i)\cdot\lambda_M^{1/2}(\mathbf{A}_{is}^{\mathrm{T}}\mathbf{A}_{is})\cdot\|\mathbf{Z}_s\|\} \\ &\quad -0{\cdot}5 d_s \lambda_M^{1/2}(\mathbf{H}_s)\cdot\|\mathbf{Z}_s\|, \end{aligned}$$

$$\leqslant \sum_{i=1}^{s-1} \{ -0{\cdot}5\, d_i \lambda_M^{-1}(\mathbf{H}_i)\mathbf{V}_i(\mathbf{Z}_i, \mathbf{t}) + d_s \lambda_M^{1/2}(\mathbf{H}_i) \cdot \lambda_M^{1/2}(\mathbf{A}_{is}^{\mathrm{T}}\mathbf{A}_{is}) \cdot \lambda_m^{-1/2}(\mathbf{H})$$

$$\cdot \mathbf{V}_s(\mathbf{Z}_s, \mathbf{t})\} - 0{\cdot}5 d_s \lambda_M^{-1}(\mathbf{H}) \cdot \mathbf{V}_s(\mathbf{Z}_s, \mathbf{t})$$

$$= -\left\{ \sum_{i=1}^{s-1} [d_i |\varpi_{ii}| \mathbf{V}_i(\mathbf{Z}_i, \mathbf{t}) - d_s |\varpi_{is}| \mathbf{V}_s(\mathbf{Z}_s, t)] + d_s |\varpi_{ss}| \mathbf{V}_s(\mathbf{Z}_s, \mathbf{t}) \right\},$$

$$= [d_1\ d_2 \cdots d_s] \bar{\mathbf{W}} \begin{bmatrix} V_1(\mathbf{Z}_1, \mathbf{t}) \\ \mathbf{V}_2(\mathbf{Z}_2, \mathbf{t}) \\ \vdots \\ \mathbf{V}_s(\mathbf{Z}_s, \mathbf{t}) \end{bmatrix}. \quad (37)$$

The aggregation matrix $\bar{\mathbf{W}}$ for the expanded rotor-bearing system is then obtained:

$$\bar{\mathbf{W}} = (\varpi_{ij})_{s \times s}$$

$$= \begin{bmatrix} -0{\cdot}5\lambda_M^{-1}(\mathbf{H}_1) & 0 & \cdots & \lambda_M^{1/2}(\mathbf{H}_1)\lambda_M^{1/2}(\mathbf{A}_{1s}^{\mathrm{T}}\mathbf{A}_{1s})\lambda_m^{-1/2}(\mathbf{H}) \\ 0 & \ddots & \vdots & \vdots \\ \vdots & \cdots & -0{\cdot}5\lambda_M^{-1}(\mathbf{H}_{s-1}) & \lambda_M^{1/2}(\mathbf{H}_{s-1})\lambda_M^{1/2}(\mathbf{A}_{s-1s}^{\mathrm{T}}\mathbf{A}_{s-1s})\lambda_m^{-1/2}(\mathbf{H}) \\ 0 & \cdots & \cdots & -0{\cdot}5\lambda_M^{-1}(\mathbf{H}) \end{bmatrix}. \quad (38)$$

Comparing equation (38) with equation (31), it can be recognized that the structure of the aggregated matrix $\bar{\mathbf{W}}$ is closely related to that of the expanded system matrix $\bar{\mathbf{A}}$.

Take vector Liapunov function as

$$\mathbf{V}(\mathbf{Z}, t) = [V_1(\mathbf{Z}_1, t)\, V_2(\mathbf{Z}_2, t) \cdots V_s(\mathbf{Z}_s, t)]^{\mathrm{T}}. \quad (39)$$

According to the stability theory and the comparison principle [10, 11], there exists

$$\dot{\mathbf{V}}(\mathbf{Z}, t) \leqslant \bar{\mathbf{W}}\mathbf{V}(\mathbf{Z}, t), \quad (40)$$

which guarantees the Liapunov stability of equation (24) if the aggregation matrix $\bar{\mathbf{W}}$ satisfies

$$\varpi_{ij} < 0 \quad i = j,\ j = 1, 2, \ldots, s. \quad (41)$$

and

$$(-1)^k \begin{vmatrix} \varpi_{11} & \cdots & \varpi_{1k} \\ \cdots & \cdots & \cdots \\ \varpi_{k1} & \cdots & \varpi_{kk} \end{vmatrix} > 0 \quad \forall k = 1, 2, \ldots, s. \quad (42)$$

We have mentioned in the Introduction that the classical decomposition– aggregation method is not so successfully applied to rotor-bearing systems because of the strong

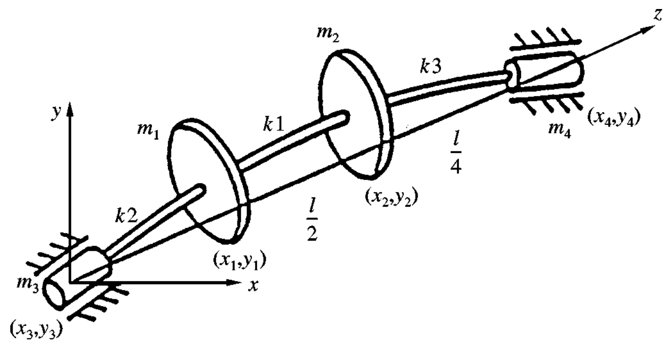


Figure 3. Multi-degree-of-freedom rotor-bearing system.

interconnection between subsystems. However, the aggregation matrix of equation (38) from the overlapping decomposition–aggregation procedure suggests that the interconnection between subsystems in expansion space has no effect on their aggregation process. The method then overcomes the difficulty of the classical method in its application to rotor-bearing system. Moreover, the aggregation matrix $\bar{\mathbf{W}}$ provides us with not only the information of every bearing system and the entire system, but also the relationship between them.

### 3.2.3. *Procedure* 7. *Stability degree*

To evaluate the degree of stability for each of the isolated subsystems in the expanded space, which include the original rotor-bearing system, the stability degree is used

$$\alpha_i = \frac{1}{\lambda_M(\mathbf{H}_i)}, \quad i = 1, 2, \ldots, s, \tag{43}$$

where Liapunov matrix $\mathbf{H}_i$ satisfies

$$\bar{\mathbf{A}}_i^{\mathrm{T}}\mathbf{H}_i + \mathbf{H}_i\bar{\mathbf{A}}_i = -\mathbf{I}_i. \tag{44}$$

It is noted that the largest dimension of matrix $\bar{\mathbf{A}}_i$ in equation (44) is merely that of the original rotor-bearing system.

## 4. NUMERICAL EXAMPLES

The single-span rotor-bearing configuration is taken as our numerical example, which is shown in Figure 3. The rotor is assumed to be elastic. The configuration of the hydrodynamic bearings are taken as: the diameter of the bearing $d = 0{\cdot}2$ m, the width/length ratio of the bearing $l/d = 0{\cdot}5$, and the clearance/radius ratio of the bearing $\psi_{min} = C_{min}/r = 0{\cdot}004$. The lamina flow of the lubrication oil in the bearing is assumed and its viscosity $\eta$ is taken as 0·0042 Pa s. The hydrodynamic lubricated bearing is assumed as the isothermal.

The governing equation of motion for this rotor-bearing system in state space takes the form

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X},$$

where the system matrix **A** denotes

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 & \mathbf{A}_{13} \\ 0 & \mathbf{A}_2 & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_3 \end{bmatrix}$$

and

$$\mathbf{A}_1 = \begin{bmatrix} -\dfrac{c_{xx}}{m_3} & -\dfrac{c_{xy}}{m_3} & -\dfrac{k_{xx}+k_2}{m_3} & -\dfrac{k_{xy}}{m_3} \\ -\dfrac{c_{yx}}{m_3} & -\dfrac{c_{yy}}{m_3} & -\dfrac{k_{yx}}{m_3} & -\dfrac{k_{yy}+k_2}{m_3} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} -\dfrac{c_{xx}}{m_4} & -\dfrac{c_{xy}}{m_4} & -\dfrac{k_{xx}+k_3}{m_4} & -\dfrac{k_{xy}}{m_4} \\ -\dfrac{c_{yx}}{m_4} & -\dfrac{c_{yy}}{m_4} & -\dfrac{k_{yx}}{m_4} & -\dfrac{k_{yy}+k_3}{m_4} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_3 = \begin{bmatrix} -\dfrac{c}{m_1} & 0 & -\dfrac{k_1+k_2}{m_1} & 0 & 0 & 0 & \dfrac{k_1}{m_1} & 0 & \\ 0 & -\dfrac{c}{m_1} & 0 & -\dfrac{k_1+k_2}{m_1} & 0 & 0 & 0 & \dfrac{k_1}{m_1} & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & \dfrac{k_2}{m_2} & 0 & -\dfrac{c}{m_2} & 0 & -\dfrac{k_1+k_2}{m_2} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{k_2}{m_2} & 0 & -\dfrac{c}{m_2} & 0 & -\dfrac{k_1+k_2}{m_2} & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \end{bmatrix},$$

$$\mathbf{A}_{13} = \begin{bmatrix} 0 & 0 & \dfrac{k_2}{m_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{k_2}{m_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_{23} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{k_3}{m_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{k_3}{m_4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_{31} = \begin{bmatrix} 0 & 0 & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & \frac{k_2}{m_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{k_3}{m_2} & 0 \\ 0 & 0 & 0 & \frac{k_3}{m_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $c_{xx}$, $c_{xy}$, $c_{yx}$, $c_{yy}$ denote the linearized bearing damping coefficients, $k_{xx}$, $k_{xy}$, $k_{yx}$, $k_{yy}$ the stiffness, and $c$ the damping acting on the rotor, $k_1$, $k_2$, $k_3$ the stiffness coefficients of the different rotor segments, and $m_1$, $m_2$, $m_3$, $m_4$ the lumped masses of the rotor.

The expanded rotor-bearing system by overlapping decomposition–aggregation is

$$\dot{\mathbf{Z}} = \bar{\mathbf{A}}\mathbf{Z},$$

where the expanded system matrix $\bar{\mathbf{A}}$ has the form

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_i & 0 & \bar{\mathbf{A}}_{1s} \\ 0 & \mathbf{A}_2 & \bar{\mathbf{A}}_{2s} \\ 0 & 0 & \mathbf{A} \end{bmatrix}, \quad \bar{\mathbf{A}}_{1s} = [0 \ \vdots \ \mathbf{A}_{13}]_{4\times 16}, \quad \bar{\mathbf{A}}_{2s} = [0 \ \vdots \ \mathbf{A}_{23}]_{4\times 16}$$

and $\mathbf{A}_1$, $\mathbf{A}_2$ denote the two bearing subsystems and $\mathbf{A}$ denotes the original system.

Table 1 shows the analytical results of the system stability. The two cylindrical bearings are assumed to bear different loading distribution, which is often the case of the multi-span rotor-bearing system. Evidently, by the proposed method, not only the stability state of the whole system can be obtained, but also the stability status of the specified subsystems. That is, the information on the two levels of the system can be obtained simultaneously. Moreover, the selection and decomposition of a subsystem depends only on the choice of the researcher in the example. This feature is of great significance.

Table 2 presents the results of the stability analyses when the two bearing subsystems are of different types, but bear on the same load. As we can see, when one of the cylindrical bearing subsystems is replaced by either the three-lobe bearing or the elliptical bearing subsystems, the stability of the system is improved. That is, both the stability threshold speed and the degree of stability of the system are increased. Since the stability performance of these bearing subsystems is, as we know, better than the cylindrical one, these results are expected. The methodology developed here provides a way of knowing the changes of the system and its subsystems at the same time. It is definitely of both theoretical and practical significance.

Table 3 provides the analytical results when the two bearing subsystems bear on different loads while the loading of the cylindrical bearing subsystem is unchanged. This example also suggests the case of multi-bearing-rotor systems. The data show that the stability of the cylindrical bearing subsystem has no changes, but along with the change of

TABLE 1

*Stability of the rotor-cylindrical bearing system*

| Speed of rotor (r.p.m.) | Stability degree of bearing subsystem 1 | Stability degree of bearing subsystem 2 | Stability degree of the system ($\times 10^{-2}$) | Eccentric ratio of bearing 1 | Eccentric ratio of bearing 2 |
|---|---|---|---|---|---|
| 1000 | 0·358 | 0·410 | 0·426 | 0·300 | 0·400 |
| 2000 | 0·247 | 0·310 | 0·751 | 0·172 | 0·250 |
| 3000 | 0·169 | 0·226 | 1·048 | 0·119 | 0·178 |
| 3500 | 0·140 | 0·190 | 0·566 | 0·103 | 0·156 |
| 3625 | 0·133 | 0·181 | 0·134 | 0·099 | 0·151 |
| 3658 | 0·131 | 0·179 | 0·017 | 0·098 | 0·150 |
| 3660 | 0·131 | 0·179 | 0·002 | 0·098 | 0·149 |
| > 3660 | Unstable | | | | |

TABLE 2

*Stability of the rotor system with different bearing types**

| System | Stability degree of bearing subsystem 1 (eccentric ratio) | Stability degree of bearing subsystem 2 (eccentric ratio) | Stability degree of system ($\times 10^{-2}$) | Stability threshold (r.p.m) |
|---|---|---|---|---|
| Cylindrical bearing ~cylindrical bearing | 0·577 (0·647) | 0·577 (0·647) | 0·190 | 3389 |
| Cylindrical bearing ~three lube bearing | 0·577 (0·647) | 1·291 (0·4331) | 0·238 | 9037 |
| Cylindrical bearing ~elliptical bearing ($m = 0{\cdot}5$) | 0·577 (0·647) | 1·143 (0·434) | 0·232 | 6152 |
| Cylindrical bearing ~elliptical bearing ($m = 3/4$) | 0·577 (0·647) | 1·206 (0·1889) | 0·238 | 15668 |

*The data of degree of stability are obtained at 2000 r.p.m.

TABLE 3

*Stability of the rotor system with different loading and bearing types**

| System | Stability degree of bearing subsystem 1 (eccentric ratio) | Stability degree of bearing subsystem 2 (eccentric ratio) | Stability degree of system ($\times 10^{-2}$) | Stability threshold (r.p.m) |
|---|---|---|---|---|
| Cylindric ~three lobe | 0·969 (0·429) | 1·662 (0·280) | 0·435 | 8242 |
| Cylindric ~elliptical ($m = 0{\cdot}5$) | 0·969 (0·429) | 1·246 (0·244) | 0·386 | 5869 |
| Cylindric ~elliptical ($m = 2/3$) | 0·969 (0·429) | 1·222 (0·107) | 0·418 | 8740 |
| Cylindric ~elliptical ($m = 3/4$) | 0·969 (0·429) | 1·264 (0·057) | 0·419 | 14121 |

*The data of degree of stability are obtained at 5500 r.p.m.

the other bearing, both the stability of the whole system and that of the replaced bearing subsystem are obviously improved. From the data, we can also distinguish the stability performance of different bearing types and know the changes and improvements of system performance by knowing the changes of the subsystems. Again, with the help of the methodology, we can analyze the stability of the entire system and subsystems, and we can also know the changes of the stability of the entire system along with the changes of its subsystems and their interconnections. The proposed method provides us with the most flexible way to investigate what we want to know. It is believed that this superiority is hardly accessible by other traditional methods.

## 5. CONCLUSIONS

The whole and the part are general attributes of nature. A rotor-bearing compound system that is overlappingly decomposed into different levels is a concrete reflection of this thought. The methodology developed in this study provides us with a powerful analysis tool to study some aspects of the performance of dynamic system. The following conclusions can be drawn:

(1) The dynamic system can be viewed as comprising two levels—the whole system and its subsystems—simultaneously by using the method proposed here if it is required in a study;
(2) With the method, the stability of the system and its specified subsystems can be studied at the same time. Information on the interconnection between the system and its subsystems can also be obtained.
(3) Theoretically, the subsystem could be decomposed from a purely mathematical point of view. In practice, the physical entity ought to be considered in the decomposition of subsystems from a system.

The overlapping decomposition–aggregation method itself is based on the general comparison principle of differential equation theory and Liapunov stability theory. Therefore, the methodology presented in this paper is adapted to non-linear problems. But, the construction of Liapunov-type functions for a general non-linear system is difficult. However, it is hopefully possible that we may find the Liapunov function for some special non-linear problems [9]. Further studies on this aspect are worthwhile.

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